Waves and Vortices in Optical and BEC Turbulence

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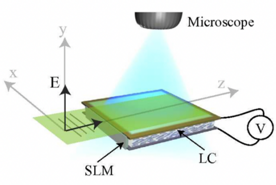
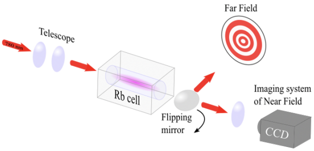
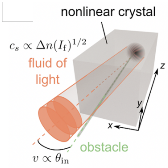
Nonlinear Optical Systems at INPHYNI







Left: Liquid Crystal Cell for 1D turbulence (Bortolozzo & Residori). Centre, right: Hot vapour (P. Azam, R. Kaiser) and photorefractive crystal (A. Eloy, M. Bellec & C. Michel) for 2D turbulence. We also plan to study 1D turbulence in optical fibers

Optical and BEC turbulence.

On the most basic level, optical and BEC turbulence is described by Gross-Pitaevskii (a.k.a. NLS) equation:

$$i\,\frac{\partial \psi}{\partial t} + \nabla^2 \psi - |\psi|^2 \psi = 0. \qquad (1)$$

where ψ is a complex scalar field.
GP equation (1) conserves two quantities with positive quadratic parts— the total number of particles,

$$N = \int |\psi(\mathbf{x}, t)|^2 d\mathbf{x}\,, \qquad (2)$$

and the total energy,

$$H = \int \left[|\nabla \psi(\mathbf{x}, t)|^2 + \frac{1}{2} |\psi(\mathbf{x}, t)|^4 \right] d\mathbf{x}\,, \qquad (3)$$

Fluid properties of the GP system

There are strong parallels and analogies between the GP model and the classical fluids, which could be understood by making the Madelung transformation: $\psi = \sqrt{\rho}\, e^{i\phi}$. After this, the GP equation (1) becomes very similar to an ideal fluid system with density ρ and velocity $\mathbf{u} = 2\nabla\phi$:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \qquad \text{mass balance} \qquad (4)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\frac{\nabla \rho^2}{\rho} + \nabla \left(2 \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \right) \qquad \text{momentum balance} \qquad (5)$$

Eq (4) it identical to the corresponding equation for classical fluids. Eq (5) is very similar to the momentum balance equation for an ideal fluid with pressure $p = \rho^\gamma,\ \gamma = 2$. An immediate consequence is that the GP system possesses fluid-like states including randomly moving vortices and waves, i.e. vortex and wave turbulence. The last term in (5) represents the only difference with the ideal fluid case; it is called "quantum pressure".

Important structures in hydro and Optical turbulence

1. Vortices: Important in hydro and Optical turbulence. They may have arbitrary continuous solenoidal vorticity fields in classical turbulence. In quantum turbulence, there are only point vortices with quantised circulation. The quantum vortices are located at points where $\psi = 0$ and the circulation $\Gamma = \oint_C \mathbf{u}(\mathbf{x})\, d\ell = 2 \oint_C \nabla\theta\, d\ell = 2[\theta]_C = \pm 4\pi$. (Multiply charged vortices are structurally unstable.)

2. Waves: Sound waves and Kelvin waves on vortex filaments are common for hydro and quantum turbulence. De Broglie (matter) waves exist in Optical systems only.

Unlike classical vortices, quantum vortices can be created and annihilated without dissipation. In doing so they emit sound. Vortex and wave components can coexist, interact and create each other. Pure vortex and pure wave turbulence arise in the strong and weak nonlinearity limits.

Weak wave turbulence

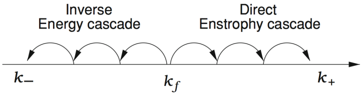
Weak wave turbulence (WWT) refers to systems with random weakly nonlinear waves. In WWT, waveaction spectrum $n_{\mathbf{k}} = (L/2\pi)^d \langle |\psi_{\mathbf{k}}|^2 \rangle$ evolves according to the wave-kinetic equation (WKE):

$$\partial_t n_{\mathbf{k}} = 4\pi \int |n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} n_{\mathbf{k}} \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_3}} - \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} \right] \times$$
$$\delta(\mathbf{k} + \mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2)\, \delta(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3, \qquad (6)$$

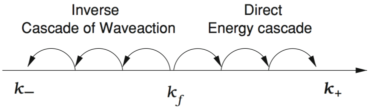
where $\omega_{\mathbf{k}} = k^2$.
Now the invariants are: $N = \int n_{\mathbf{k}} d\mathbf{k}$ and $E = \int k^2 n_{\mathbf{k}} d\mathbf{k}$.

Dual cascades

2D turbulence



Weak wave turbulence



Standard (Fjortoft'1953) argument in 2D turbulence predicts a dual cascade behaviour: energy cascades to low wavenumbers while enstrophy cascades to high wavenumbers. Similar argument in WT predicts a forward cascade of energy and an inverse cascade of waveaction (particles).

- Direct E-cascade: "evaporation".
- Inverse N-cascade: Non-equilibrium condensation.

Dual cascades in BEC

Yu. Lvov et al. / Physica D 184 (2003) 333–351

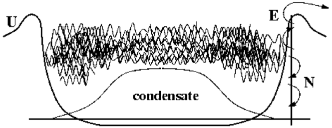
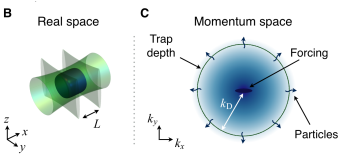


Fig. 1. Turbulent cascades of energy E and particle number N.

BEC Turbulence experiment of Navon et al.'2018.



- Direct E-cascade: "evaporation".
- Inverse N-cascade: Non-equilibrium condensation.

Kolmogorov-Zakharov spectra in the GP model

Stationary Kolmogorov-Zakharov (KZ) spectra $n_k \sim k^{\nu}$ are solutions of WKE corresponding to the energy and the particle cascades:

$$\nu_E = -d, \quad \curvearrowright\curvearrowright\curvearrowright$$

and

$$\nu_N = -d + 2/3, \quad \curvearrowleft\curvearrowleft\curvearrowleft .$$

KZ spectra are only meaningful if they are *local*, i.e. when the collision integral in the original kinetic equation converges.
In 3D ($d = 3$) the inverse $\mathcal{N}$-cascade spectrum is local, whereas the the direct E-cascade spectrum is log-divergent at $k \to 0$. As usual, the log-divergence can be remedied by a log-correction,

$$n_k \sim [\ln(k/k_f)]^{-1/3} \, k^{\nu_E},$$

KZ solutions of the GP system cont'd

The 2D case ($d = 2$) appears to be even more tricky. Formally the $\mathcal{N}$-cascade spectrum is local, but the $\mathcal{N}$-flux is positive, in contradiction with the Fjørtoft's argument. For the E-cascade spectrum, the exponent ν_E coincides with the one of the thermodynamic E-equipartition spectrum. As a result, the KZ spectra are not realisable in the 2D GP turbulence. Instead, "warm cascade" states are observed where the E and N k-space fluxes are on background of a thermalised background.

The 1D case is described by a six-wave process because there are no nontrivial four-wave ($2 \to 2$) resonances in 1D systems with $\omega_k \sim k^2$.

Direct and inverse cascades in 2d GPE

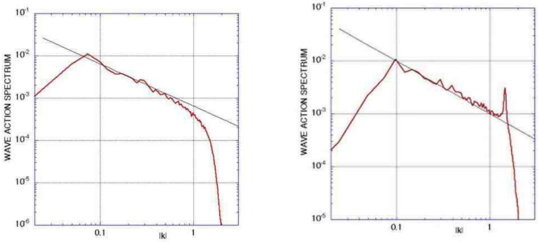


Figure: SN & M. Onorato (2006)

Both direct and inverse cascades are "warm": their spectra are thermal equipartition of energy with small corrections to accommodate E and N fluxes.

Evolving 2d GP turbulence

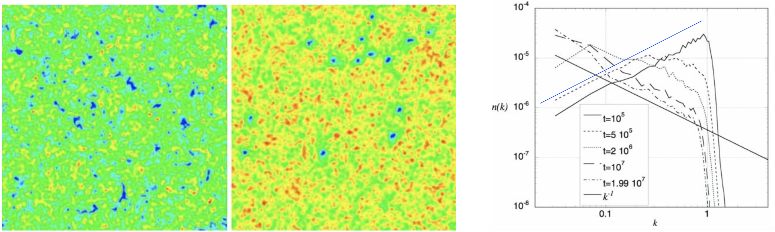


Figure: SN & M. Onorato (2007)

Evolution scenario: 4-wave WT of de-Broglie waves → hydrodynamics of point vortices → 3-wave WT of Bogolubov sound. Presently we are studying self-similarity of the evolving spectrum.

1D optical turbulence in liquid crystal (Laurie, Bortolozzo, Nazarenko & Residori, 2009).

Light affects orientation of the LC molecules and, therefore, the refractive index.

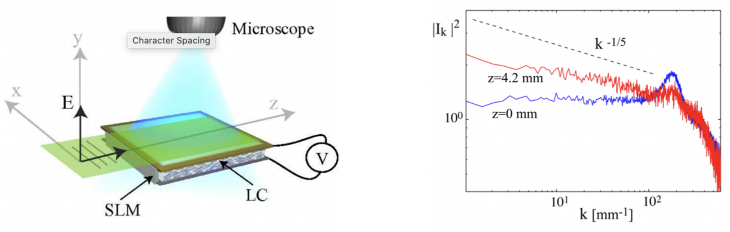
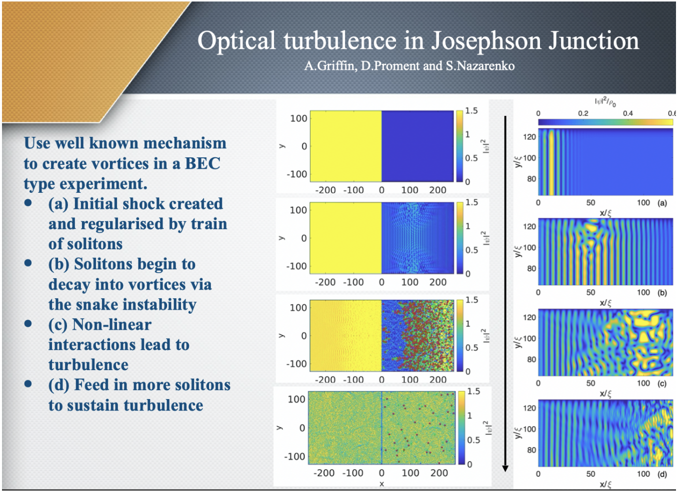


Figure: Left: LC cell setup; Right: the light intensity spectrum versus the WTT prediction

2D turbulence LC-based experiment is planned.

2D optical turbulence



Two 2D turbulence experiments are underway at INPHYNI using atomic vapour and a photo-refractive crystal.

2D optical vortices in atomic vapour (P. Azam, A. Griffin, R. Kaiser, SN, 2020)

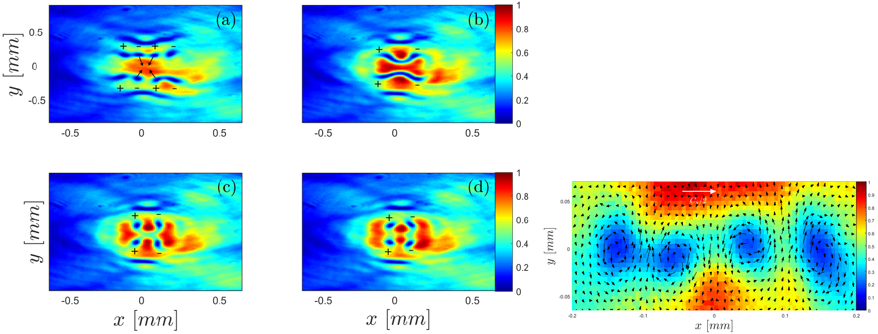


Figure: Vortices in atomic vapour

Initial dark soliton breaks via a snaking instability into vortices

2D optical vortices in atomic vapour (A. Eloy, M. Bellec, A. Griffin, C. Michel, SN, 2021)

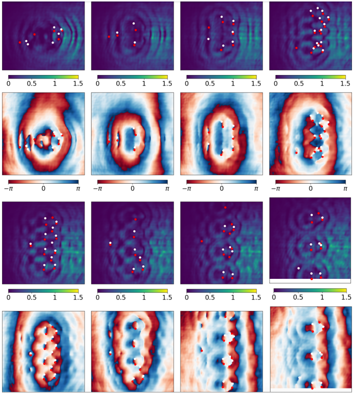


Figure: Vortices in photo-refractive crystal: flow past an a grid

Vortex production is maximized for an optimal spacing between obstacles.

2D optical turbulence as a model for galaxy formation

Wave Turbulence in Self-Gravitating Bose Gases

J. Skipp, V. L'vov, S. Nazarenko

Schrödinger-Newton Equation

$$i\partial_t\psi + \nabla^2\psi - \psi V = 0$$
$$\nabla^2 V = Gm|\psi|^2$$

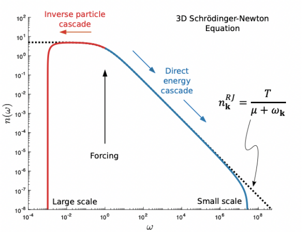
3D: "Fuzzy Dark Matter" of self-gravitating bosons.
2D: Optical systems with nonlocal nonlinearity.

Wave Kinetic Eq: $\partial_t n_{\mathbf{k}} = 4\pi \int \left|W^{12}_{3\mathbf{k}}\right|^2 \delta^{12}_{3\mathbf{k}} \delta\left(\omega^{12}_{3\mathbf{k}}\right) n_1 n_2 n_3 n_{\mathbf{k}} \left[\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_{\mathbf{k}}}\right] d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$

Particles cascade to large-scale, energy cascades to small-scale.

Scaling solutions k^{-x} predict wrong cascade directions.

Dual cascade realised by quasi-thermal spectrum $n^{RJ}_{\mathbf{k}} = \frac{T}{\mu + \omega_{\mathbf{k}}}$



J. Skipp, et al. arXiv:2003.0558 (2020)

A 2D turbulence experiment with nonlocal nonlinearity is planned at INPHYNI using a liquid crystal.

Evolving weak 3D BEC turbulence

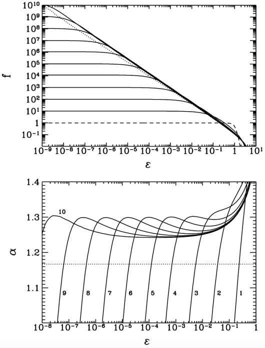
Isotropic wave-kinetic equation after integrating out the angles and passing from the wave number to the frequency variable:

$$\frac{d}{dt}n_\omega = \omega^{-1/2}\int \min\left(\sqrt{\omega},\sqrt{\omega_1},\sqrt{\omega_2},\sqrt{\omega_3}\right) n_\omega n_1 n_2 n_3 \left(n_\omega^{-1}+n_1^{-1}-n_2^{-1}-n_3^{-1}\right)\delta(\omega+\omega_1-\omega_2-\omega_3)d\omega_1 d\omega_2 d\omega_3. \quad (7)$$

where $\omega = k^2$ is the wave frequency and $n_\omega(t) \sim \langle|\psi_k|^2\rangle$ is the spectrum.

Self-similar evolution in the inverse cascade range

Non-equilibrium condensation process.
(Semikoz and Tkachev 1995, Lacaze et al 2001)



Solution "blows up" in finite time t^*. Shortly before t^* they reported $n = \omega^{-x^*}$
$x^* = 1.23 > 1.16 = x_{KZ}$. Thermodynamic $n = 1/\omega$ is observed after t^*.

B. Semisalov, V. Grebenev, S. Medvedev and SN (2021).
Let us seek solution of the WKE in a similarity form $n_\omega = \tau^a f(\eta)$, where $\eta = \omega\tau^{-b}, b = a - 1/2 > 0, \tau = t^* - t$. Then WKE can be rewritten as

$$xf + \eta f' = \frac{1}{b} St[f], \quad x = \frac{a}{b} \tag{8}$$

Self-similarity of the 2nd type: a and b cannot be found from a conservation law, but are solutions of a nonlinear eigenvalue problem.
Boundary conditions:
(1) $f(\eta) \to \eta^x$ for $\eta \to \infty$.
(2) $f(\eta) \to$ const for $\eta \to 0$. This BC follows from previous numerics, but it can be rigorously justified.

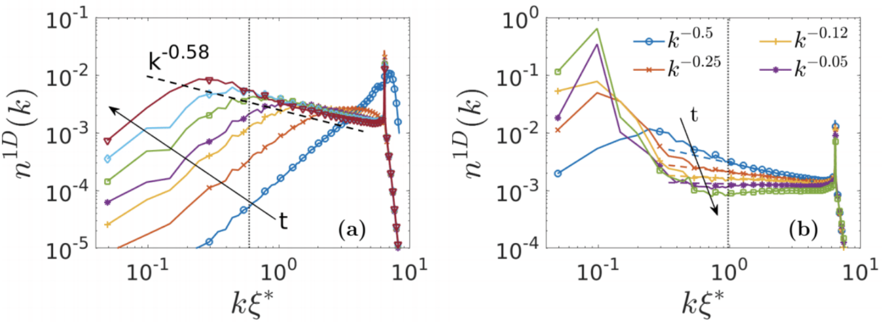
Self-similar solution of WKE

$$xf + \eta f' = \frac{1}{b} St[f] \tag{9}$$

Nonlinear eigenvalue problem: find x for which the following boundary conditions are satisfied simultaneously.
(1) $f(\eta) \to \eta^{-x}$ for $\eta \to \infty$. (2) $f(\eta) \to$const for $\eta \to 0$.
It is much harder to solve the equation for $f(\eta)$ than to solve WKE for evolving $n(k, t)$.
Relaxation of iterations. Best accuracy in terms of the sup-norm of the relative mismatch: 4.7% for $x^* = 1.22$.

3D BEC turbulence study via 512^3 DNS of the GP equation.

V. Shukla and SN (2020).



The exponent 0.58 is close to the KE prediction of 0.46 (corresponding to $x^* = 1.23$). Late time: Condensate peak at low k and thermal energy equipartition at high k.

Fourth-order differential approximation to four-wave WKE

S. Thalabard, S. Medvedev and V. Grebenev (2021).

$$\partial_t n = \omega^{1-d/2} \frac{\partial^2}{\partial \omega^2} \left(\omega^s n^4 \frac{\partial^2}{\partial \omega^2} \left(\frac{1}{n} \right) \right). \tag{10}$$

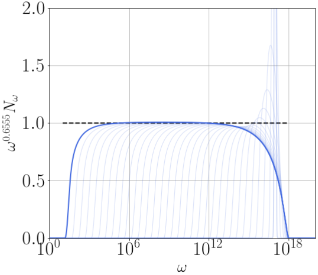
Here s depends on the particular 4-wave system, e.g. BEC, gravitational waves,...

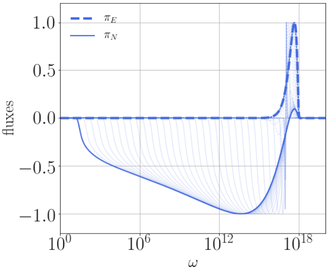
This can be transformed into a 4D autonomous dynamical system. The nonlinear eigenvalue problem is to find x for which the following boundary conditions are satisfied:

(1) power law with exponent x for large frequencies.

(2) sharp front propagating to the left at which there is no dissipation (flux).

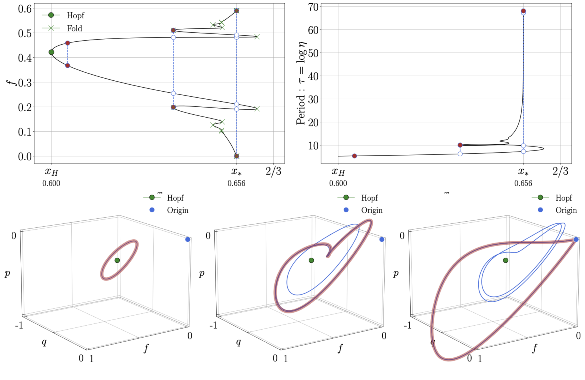
Numerics of the initial value problem (Gravitational Wave turbulence)





We see $x^* < 2/3$ which is the KZ value.

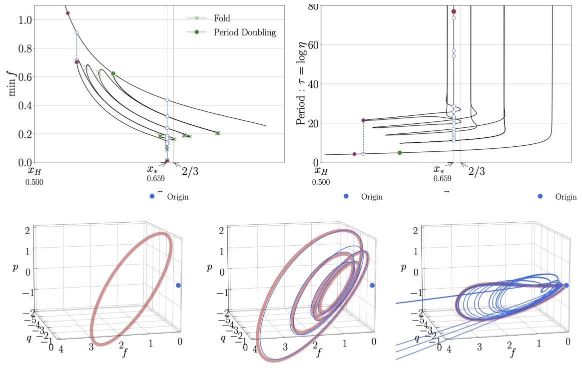
4D dynamical system (Gravitational Wave turbulence)



(c) $x = 0.605$ (d) $x = 0.636$ (e) $x = 0.656$

In 4D the solution to the nonlinear eigenvalue problem is again given by a global biffurcation! Hopf $\rightarrow$ cycle $\rightarrow$ cycles via fold bifurcations (up to 5!) $\rightarrow$ homoclinic cycle (with 2 other cycles hovering on background).

4D dynamical system (3D BEC turbulence)



(c) $x = 0.540$ (d) $x = 0.551$ (e) $x = 0.659$

Similar to the GW case but Hopf moves to infinity and there are more crazy cycle bifurcations en route to the Homoclinic cycle

Summary

- Nonlinear chaotic motions of waves and vortices optical media have properties of turbulence, e.g. forward and inverse cascades.
- Inverse cascade of the waveaction/particles is a process of nonequilibrium condensation. Direct cascade of energy is a cooling process.
- Optical turbulence allows to model important processes in quantum and astropysical systems, including dark matter and galaxy formation.
- 2D optical turbulence has not been yet implemented yet, but the work is underway. 1D turbulence was implemented in LC and fibers. 3D turbulence - in BEC.
- Major remaining theoretical problem: develop a kinetic description of the vortex "gas" and incorporate it into the WT theory.